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HERMITIAN OPERATORS AND ONE-PARAMETER GROUPS OF ISOMETRIES IN HARDY SPACES

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ABSTRACT. Call an operator A with domain and range in a complex Banach space X *hermitian* if and only if iA generates a strongly continuous one-parameter group of isometries on X. Hermitian operators in the Hardy spaces of the disc ($H^p, 1 \leq p \leq \infty$) are investigated, and the following results, in particular, are obtained. For $1 \leq p \leq \infty, p \neq 2$, the bounded hermitian operators on H^p are precisely the trivial ones–i.e., the real scalar multiples of the identity operator. Furthermore, as pointed out to the authors by L. A. Rubel, there are no unbounded hermitian operators in H^∞. To each unbounded hermitian operator in the space H^p, $1 \leq p < \infty$, $p \neq 2$, there corresponds a uniquely determined one-parameter group of conformal maps of the open unit disc onto itself. Such unbounded operators are classified into three mutually exclusive types, an operator's type depending on the nature of the common fixed points of the associated group of conformal maps. The hermitian operators falling into the classification termed "type (i)" have compact resolvent function and one-dimensional eigenmanifolds which collectively span a dense linear manifold in H^p.

0. **Introduction.** A natural way to extend the notion of bounded hermitian operator from Hilbert space to the context of an arbitrary Banach space X (we use complex scalars throughout) is to call a bounded operator A from X to X hermitian if and only if $\exp(itA)$ is an isometry for each $t \in \mathbf{R}$ (where $\mathbf{R}$ is the field of real numbers and "exp" stands for "exponential of"). The definition of a bounded hermitian operator on X has an obvious generalization. Specifically, we shall call a (not necessarily bounded) linear transformation B with domain and range contained in X a hermitian operator if and only if iB generates a strongly continuous one-parameter group of isometries on X (we employ the definitions and terminology of [2, Chapter VIII] for semigroups of operators). In a Hilbert space $\mathcal{K}$ the hermitian operators, as defined in the preceding sentence, are of course precisely the selfadjoint linear transformations with domain and range in $\mathcal{K}$. The above notion of bounded hermitian operator on a Banach space was first formulated and investigated (from formally separate, but convergent viewpoints) in [13] and [10]. Unbounded hermitian operators in the Banach space setting have been considered (by implication) from the standpoint of their analogy with Hilbert space concepts in [11]. For example, it is obvious from Theorem (3.1) of [11] that a (bounded or unbounded) hermitian operator B with domain $\mathfrak{D}(B)$ in

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X has the property that $x^*Bx \in \mathbf{R}$, whenever $x \in \mathfrak{D}(B)$, $x^* \in X^*$ (= the normed conjugate of X) with $\|x^*\| = \|x\|$ and $x^*(x) = \|x\|^2$.

Our purpose is to investigate hermitian operators in the Hardy spaces H^p(of the disc) for $1 \leq p \leq \infty$. We rely heavily on Forelli's characterization [4, Theorem 2] of the linear isometries of H^p onto H^p for $1 \leq p < \infty, p \neq 2$, which extends the result in [9] for the case $p = 1$. Except as otherwise noted, for terminology and notation, as well as basic facts, we follow [8] regarding Hardy spaces and [2] regarding general Banach space concepts.

Throughout what follows, the term "operator in X" means "linear transformation with domain and range in X." We denote the complex plane by $\mathbf{C}$, the open unit disc $\{z \in \mathbf{C}: |z| < 1\}$ by D, the unit circle $\{z \in \mathbf{C}: |z| = 1\}$ by C, and normalized Lebesgue measure on C by σ. Thus, for $1 \leq p \leq \infty$, H^p may be regarded as $\{f \in L^p(\sigma): \int f(z)z^n d\sigma(z) = 0$, for $n = 1, 2, \ldots\}$ or as the space of analytic functions f on D for which the functions $f_r(\theta) = f(re^{i\theta})$, $0 < r < 1$, are uniformly bounded in $L^p(\sigma)$. We shall deal with H^p in the latter form (except for a brief time in §3, where the switch to its first description above will be clear from the context).

Application of Forelli's characterization of surjective isometries of H^p quickly leads to groups (under composition of mappings) of univalent analytic maps of D onto D (sometimes called Möbius transformations of the disc). For convenience, groups of Möbius transformations of D are considered beforehand in §1. §2 deals with hermitian operators and one-parameter groups of isometries in the spaces H^p, $1 \leq p \leq \infty$. Since in the case of the Hilbert space H^2 the development of these topics is well known, the case $p = 2$ is considered in §2 only to the extent of dealing with analogues in H^2 of groups of isometries on the spaces H^p for $1 \leq p < \infty$, $p \neq 2$. As might be expected, the space H^2 differs markedly from the other Hardy spaces in the degree of abundance of hermitian operators. In fact, for $1 \leq p \leq \infty$, $p \neq 2$, the bounded hermitian operators on H^p are precisely the trivial ones–i.e., the operators rI, for $r \in \mathbf{R}$, where I denotes the identity operator on H^p (see Theorem (2.8) below). The space H^∞ distinguishes itself by having no unbounded hermitian operators with domain in it (see Theorem (2.7)), whereas the discussion in (2.9) below illustrates the fact that in the spaces H^p, $1 \leq p < \infty$, unbounded hermitian operators always exist. §2 also contains a classification of the unbounded hermitian operators in H^p, $1 \leq p < \infty, p \neq 2$, into three mutually exclusive types (labeled types (i), (ii), and (iii)). §3 deals with the spectral analysis of hermitian operators of type (i). It turns out that for such operators the values of the resolvent are always compact operators. Moreover, a hermitian operator of type (i) is completely determined by its spectrum and the collection of all its eigenvectors.

1. **Groups of Möbius transformations of the disc.** In this section we shall be concerned with univalent analytic maps of D onto D. These are precisely [5, p. 226] the functions of the form

$$\phi(z) = \lambda(z - z_0)(1 - \bar{z}_0 z)^{-1},$$
$$\text{where } \lambda,\ z_0 \text{ are constants with } |\lambda| = 1,\ |z_0| < 1. \tag{1.1}$$

These functions (above called Möbius transformations of D) clearly constitute a group (denoted $\mathcal{M}$) under composition of maps. We remark that functions defined for all z in the extended complex plane $\mathbf{C}_e$ by an equation of the form (1.1) have restrictions to C, D, and $\bar{D}$ (= the closure of D) which are, in each instance, uniquely expressible on their domain of definition in the form (1.1), and which map their domain of definition one-to-one onto itself. Thus the groups of functions of the form (1.1) on, respectively, $\mathbf{C}_e$, C, D, and $\bar{D}$ are naturally isomorphic. While we shall deal primarily with Möbius transformations of D, it will sometimes be convenient to make use of one of the natural isomorphisms, and to identify tacitly corresponding group elements.

Let us recall some basic facts (also recorded in [12, Lemma 5]) about the fixed points of a Möbius transformation of D. If ϕ has the form (1.1) with $z_0 \neq 0$, then the fixed points in $\mathbf{C}_e$ of ϕ are precisely the zeros of the quadratic polynomial $q(z) = \bar{z}_0 z^2 + (\lambda - 1)z - \lambda z_0$. Since the product of the zeros of q is unimodular, one of the following situations must occur: (i) q has a root outside $\bar{D}$ and a root in D; (ii) q has a double root belonging to C; (iii) q has two distinct roots belonging to C. By appropriate choice of ϕ, each of these situations can be realized. Also, since, for $z \neq 0$, $q(z) = -\lambda z^2 \overline{q(\bar{z}^{-1})}$, we have that if τ is a fixed point of ϕ, then so is $\bar{\tau}^{-1}$. Of course, if we take $z_0 = 0$ in (1.1) then $\phi(z) \equiv \lambda z$, and so ϕ is the identity, or has 0 and ∞ as its fixed points.

Throughout the remainder of this section, let $\{\phi_t\}$, $t \in \mathbf{R}$, be a one-parameter group of Möbius transformations of D—i.e., $t \mapsto \phi_t$ is a homomorphism from the additive group of real numbers into $\mathcal{M}$. In this section (but only in this section) we shall exclude the trivial case by making the added assumption that $\{\phi_t\}$ is a nonconstant group—i.e. there is an $s \in \mathbf{R}$ such that ϕ_s is not the identity map (s will also be fixed in this section). Let

$$\phi_t(z) = a_t(z - b_t)(1 - \bar{b}_t z)^{-1} \quad \text{for } z \in D,\ t \in \mathbf{R}, \tag{1.2}$$

where a_t and b_t are constants (uniquely determined by t) such that $|a_t| = 1$ and $|b_t| < 1$.

(1.3) **Theorem.** *If, for each fixed $z \in D$, the function $t \mapsto \phi_t(z)$ is continuous on* $\mathbf{R}$, *then the functions $t \mapsto a_t$ and $t \mapsto b_t$ belong to $C^\infty(\mathbf{R})$ (i.e., have derivatives of every order on* $\mathbf{R}$).

Proof. $\phi_{-t}(0)$ is continuous in t, but for each $t \in \mathbf{R}$, $\phi_{-t}(0) = (\phi_t)^{-1}(0) = b_t$. Let u be any real number. For each $z \in D$ distinct from b_u, $\lim_{t \to u} \bar{a}_t \phi_t(z) = \bar{a}_u \phi_u(z)$. Since $\phi_u(z) \neq 0$ and $\lim_{t \to u} \phi_t(z) = \phi_u(z)$, the function $t \mapsto a_t$ is continuous at u. By [6, Theorem (2.6)] the functions $t \mapsto a_t$ and $t \mapsto b_t$ have real and imaginary parts which are real analytic on $\mathbf{R}$.

(1.4) **Corollary.** *$\phi_t(z)$ is continuous in t for each $z \in D$ if and only if it is continuous in t for each $z \in \overline{D}$.*

Proof. If $\phi_t(z)$ is continuous in t for each $z \in D$, then $t \mapsto a_t$ and $t \mapsto b_t$ are continuous on $\mathbf{R}$.

The treatment in §2 will rely heavily on the nature of the common fixed points of the functions ϕ_t. For ease of reference, we shall briefly discuss here this aspect of the one-parameter group $\{\phi_t\}$ in a form convenient for our purposes. Observe that if α is a fixed point in $\mathbf{C}_e$ of ϕ_s, then for each $t \in \mathbf{R}$, $\phi_t(\alpha) = \phi_t(\phi_s(\alpha)) = \phi_s(\phi_t(\alpha))$; so $\phi_t(\alpha)$ is a fixed point of ϕ_s. Suppose first that $\overline{D}$ contains precisely one fixed point τ of ϕ_s. Since ϕ_s has no fixed point in $\overline{D}$ except τ, $\phi_t(\tau) = \tau$ for all $t \in \mathbf{R}$. If $\tau \in D$, we observe that ϕ_s has exactly one other fixed point γ in $\mathbf{C}_e$, and $\gamma \notin \overline{D}$. An argument similar to the one just used shows that γ is also a fixed point of every ϕ_t for $t \in \mathbf{R}$. Suppose next that ϕ_s has exactly two fixed points in $\mathbf{C}_e$, τ_1 and τ_2, both in C. In this case, we make the additional hypothesis that the equivalent continuity conditions of Corollary (1.4) hold. For each $t \in \mathbf{R}$, ϕ_t must leave each of τ_1 and τ_2 fixed or permute them. By continuity each of the disjoint sets $\{t \in \mathbf{R}\colon \phi_t(\tau_k) = \tau_k, k = 1,2\}$ and $\{t \in \mathbf{R}\colon \phi_t(\tau_1) = \tau_2, \phi_t(\tau_2) = \tau_1\}$ is closed. Using the connectedness of $\mathbf{R}$ and the fact that ϕ_0 is the identity map, we get that every ϕ_t, $t \in \mathbf{R}$, has τ_1 and τ_2 as fixed points. We have shown:

(1.5) **Proposition.** *Let $\{\phi_t\}$, $t \in \mathbf{R}$, be a nonconstant one-parameter group of Möbius transformations of the disc D such that for each $z \in D$, $\phi_t(z)$ is a continuous function of t. Then the set $\mathcal{S}$ of common fixed points in $\mathbf{C}_e$ of the functions ϕ_t, $t \in \mathbf{R}$, must be one of the following*:

(i) *a doubleton subset of $\mathbf{C}_e$ consisting of a point τ in D and of $\bar{\tau}^{-1}$ (the latter taken to be ∞ if $\tau = 0$),*

(ii) *a singleton subset of C, or*

(iii) *a doubleton subset of C.*

If u is any real number such that ϕ_u is not the identity function, then $\mathcal{S}$ coincides with the set of all fixed points of ϕ_u in $\mathbf{C}_e$.

Definition. The group $\{\phi_t\}$, $t \in \mathbf{R}$, of Proposition (1.5) will be called a group of type (i), (ii), or (iii) according as (i), (ii), or (iii) of the mutually exclusive descriptions in Proposition (1.5) for its set of common fixed points in $\mathbf{C}_e$ is valid.

Remark. Let $\mathcal{G}$ be a family of continuous mappings of $\overline{D}$ into itself, with each member of $\mathcal{G}$ analytic in D. If $\mathcal{G}$ is a commutative family (with respect to composition of mappings), then a result of Shields [12, Theorem 1] states that the members of $\mathcal{G}$ have a common fixed point in $\overline{D}$. Shields' result is, of course, applicable in the context of Proposition (1.5), but the latter's stronger hypotheses afford stronger conclusions useful in the formulations in the present paper. In this connection, however, it should be pointed out that part of the proof of [12, Theorem 1] makes strong contact with the discussion we used to establish Proposition (1.5).

By applying Theorem (1.3) to the group $\{\phi_t\}$ of Proposition (1.5), we see that for each ordered pair (r,w) with $r \in \mathbf{R}$, $w \in \mathbf{C}$, and w not a pole of $\phi_r(\cdot)$, both partial derivatives $\partial\phi_t(z)/\partial t$ and $\partial\phi_t(z)/\partial z$ exist at (r,w). It will be convenient to rewrite $\phi_t(z)$ as $\phi(t,z)$ and the respective values of these partial derivatives (at (r,w)) as $\phi_1(r,w)$ and $\phi_2(r,w)$. Notice in particular that for each $z \in \mathbf{C}$ there is a corresponding $\delta(z) > 0$ such that for each $t \in \mathbf{R}$ with $|t| < \delta(z)$, $\phi_1(t,z)$ and $\phi_2(t,z)$ exist. Let us also rewrite the functions $t \mapsto a_t$ and $t \mapsto b_t$ of (1.2) as, respectively, $a(\cdot)$ and $b(\cdot)$. Their respective derivatives on $\mathbf{R}$ will be denoted $a'(\cdot)$ and $b'(\cdot)$. An easy computation shows that

$$(1.6) \qquad \phi_1(0,z) = \overline{b'(0)}z^2 + a'(0)z - b'(0), \quad \text{for all } z \in \mathbf{C}.$$

We shall call $\phi_1(0,\cdot)$ the invariance polynomial of the group $\{\phi_t\}$. This terminology is an outgrowth of the next result.

(1.7) **Theorem.** *Let* $\{\phi_t\}$, $t \in \mathbf{R}$, *and* $\mathcal{S}$ *be as in Proposition* (1.5). *Then* $\mathcal{S} \cap \mathbf{C}$ *is the set of zeros in* $\mathbf{C}$ *of* $\phi_1(0,\cdot)$.

Proof. If $w \in \mathcal{S} \cap \mathbf{C}$, then $\phi(t,w) - w = 0$ for all real t. Taking derivatives at $t = 0$ gives $\phi_1(0,w) = 0$. Conversely suppose $\phi_1(0,w) = 0$. We know that there is a $\delta > 0$ such that $\phi(\cdot,w)$ takes values in $\mathbf{C}$ and is differentiable on the open interval $(-\delta,\delta)$. Define $g\colon (-\delta,\delta) \to \mathbf{C}$ by $g(t) = \phi(t,w) - w$ for $t \in (-\delta,\delta)$. Note that $g(0) = g'(0) = 0$. If t, $u \in (-\delta/2,\delta/2)$, then $g(u+t) = \phi(u+t,w) - w$. We know that $\phi_t(w) \in \mathbf{C}$, and we also have the equation $\phi(u+t,w) = \phi(u,\phi_t(w))$. Consequently $\phi_t(w)$ is not a pole of $\phi_u(\cdot)$, and so $\phi_1(u,\phi_t(w))$ exists. Moreover, $g(u+t) = \phi(u,\phi_t(w)) - w$. If we hold t fixed and differentiate this last equation with respect to u, we get $g'(u+t) = \phi_1(u,\phi_t(w))$ for t, $u \in (-\delta/2, \delta/2)$. Setting $u = 0$ in the equation just obtained gives $g'(t) = \phi_1(0,\phi_t(w)) = \phi_1(0,g(t)+w)$ for $t \in (-\delta/2,\delta/2)$. Since w is fixed and the invariance polynomial does not have a degree greater than 2, we conclude that $g' = \alpha g^2 + \beta g + \gamma$ on $(-\delta/2,\delta/2)$, where α, β, and γ are certain constants. But $g(0) = g'(0) = 0$; so γ is 0. In summary we have

$$(1.8) \qquad g' = \alpha g^2 + \beta g \quad \text{on } (-\delta/2,\delta/2), \text{ and } g(0) = 0.$$

However, it is completely elementary (as, for example, in [1, pp. 223, 228]) that the differential equation $y' = \alpha y^2 + \beta y$ with initial condition $y(0) = 0$ has a unique complex-valued solution in some closed interval $[-\lambda,\lambda]$, where $0 < \lambda < \delta/2$. Since the zero function is obviously a solution, we have from uniqueness and (1.8) that g vanishes identically on $[-\lambda,\lambda]$. Thus $\phi_t(w) = w$ for $|t| \le \lambda$. However, if t_0 is any real number, then there is a positive integer n_0 such that $|t_0/n_0| \le \lambda$. Since ϕ_{t_0} is the n_0-fold iterate of ϕ_{t_0/n_0}, we can conclude that $\phi_{t_0}(w) = w$. Hence $w \in \mathcal{S}$.

It is an obvious consequence of Theorem (1.7) that the invariance polynomial of the group $\{\phi_t\}$ of that theorem cannot be the zero polynomial.

(1.9) **Examples.** It is easy to verify that if in (1.2) we take $a_t = (2 + it) \cdot (2 - it)^{-1}$ and $b_t = it(2 + it)^{-1}$, then the requirements on $|a_t|$ and $|b_t|$ are met, and a group of type (ii) is defined with $\mathcal{S} = \{1\}$. On the other hand, taking $a_t = 1$ and $b_t = (1 - e^t)(1 + e^t)^{-1}$ defines a group of type (iii) with $\mathcal{S} = \{1, -1\}$. Groups of type (i) have a nice characterization which we describe in the next theorem.

(1.10) **Theorem.** *Let $\{\phi_t\}$, $t \in \mathbf{R}$, be a nonconstant one-parameter group of Möbius transformations of D satisfying the equivalent continuity conditions of Corollary* (1.4). *If $\{\phi_t\}$, $t \in \mathbf{R}$, is of type* (i), *there are uniquely determined (by $\{\phi_t\}$) constants τ and c, the former in D and the latter real and nonzero, such that the parameters a_t and b_t of* (1.2) *are given for all $t \in \mathbf{R}$ by*

$$(1.11) \qquad \begin{aligned} a_t &= (|\tau|^2 - e^{ict})(|\tau|^2 e^{ict} - 1)^{-1}, \\ b_t &= \tau(e^{ict} - 1)(e^{ict} - |\tau|^2)^{-1}. \end{aligned}$$

The constant τ in (1.11) *is in fact the unique element of D which is left fixed by all ϕ_t for $t \in \mathbf{R}$. The constant c in* (1.11) *satisfies and is determined by the equation $e^{ict} z = (\psi \circ \phi_t \circ \psi)(z)$, for $t \in \mathbf{R}$ and $z \in D$, where "$\circ$" denotes composition and $\psi \in \mathcal{M}$ is given by*

$$(1.12) \qquad \psi(z) = (z - \tau)(\bar{\tau} z - 1)^{-1} \text{ for } z \in D.$$

Conversely, if τ is any element of D and c is any nonzero real number, the parameters a_t and b_t defined for each $t \in \mathbf{R}$ by (1.11) *have moduli satisfying the requirements in* (1.2), *and define by the formula in* (1.2) *a one-parameter group $\{\phi_t\}$ of type* (i).

Proof. Let τ be the common fixed point in D of the given group of type (i), $\{\phi_t\}$. Define $\psi \in \mathcal{M}$ by (1.12). Clearly ψ permutes τ and 0; hence $\psi \circ \psi$ has 0 and τ as fixed points, and so must be the identity map. Let $\rho_t = \psi \circ \phi_t \circ \psi$, for each $t \in \mathbf{R}$. $\{\rho_t\}$, $t \in \mathbf{R}$, is obviously a one-parameter group of Möbius transformations of D, and the continuity assumption imposed on $\{\phi_t\}$ implies that likewise $\rho_t(z)$ is continuous in t for each $z \in D$. Moreover, 0 is a common fixed point of the functions ρ_t for $t \in \mathbf{R}$. Consequently for each $t \in \mathbf{R}$ there is a unique unimodular complex number μ_t such that $\rho_t(z) = \mu_t z$ for $z \in D$. The properties of $\{\rho_t\}$ guarantee easily that $\{\mu_t\}$, $t \in \mathbf{R}$, is a continuous one-parameter group in C. Hence there is a real number c such that $\mu_t = e^{ict}$ for all $t \in \mathbf{R}$. The definition of $\{\rho_t\}$, $t \in \mathbf{R}$, insures that $\{\rho_t\}$ is not a constant group (because $\{\phi_t\}$ is not). Hence $c \neq 0$. Since $\phi_t = \psi \circ \rho_t \circ \psi$ for $t \in \mathbf{R}$, straightforward computations prove the assertion about the existence of constants τ and c. We remark in passing that the existence proof also shows that $\{\phi_t\}$, $t \in \mathbf{R}$, is periodic of period $2\pi c^{-1}$. It will be convenient at this juncture to defer the proof of the uniqueness assertions and take up the converse part of the theorem instead. Suppose then that τ and c are given, with τ in D and c real and nonzero. Define a_t and b_t for each $t \in \mathbf{R}$ by (1.11). Clearly $|a_t| \equiv 1$. Since b_t is a periodic function of t, it is easy to see by elementary means that the modulus of b_t has a maximum on $\mathbf{R}$ equal to $2|\tau|(1 + |\tau|^2)^{-1} < 1$. Thus for each t in $\mathbf{R}$ we can and do define a function ϕ_t in $\mathscr{M}$ by the equation in (1.2). We also define ψ in $\mathscr{M}$ by (1.12), using the given τ. Finally, let $\rho_t(z) = e^{ict}z$, for $t \in \mathbf{R}$, $z \in D$. By reversing the straightforward computations mentioned at the end of the existence proof we get (in terms of the present ψ and $\{\rho_t\}$) $\phi_t = \psi \circ \rho_t \circ \psi$ for all $t \in \mathbf{R}$. Since $\{\rho_t\}$, $t \in \mathbf{R}$, is obviously a one-parameter group of Möbius transformations of D, so is $\{\phi_t\}$, $t \in \mathbf{R}$. Since $c \neq 0$, $\{\rho_t\}$ (and hence $\{\phi_t\}$) is not constant. It is obvious from the definition of the functions ϕ_t that $\phi_t(z)$ is continuous in t for $z \in D$, and that $\phi_t(\tau) = \tau$ for all t in $\mathbf{R}$. It follows that $\{\phi_t\}$, $t \in \mathbf{R}$, is a group of type (i), and that τ is the unique point of D left fixed by all the functions ϕ_t for t in $\mathbf{R}$. The uniqueness assertions are an immediate consequence of the observation that if a given group of type (i) is represented in the form (1.11), with c real and nonzero and τ in D, then the given group is the group $\{\phi_t\}$ constructed from c and τ in the converse proof just concluded, and the discussion in that proof applies to it.

Given a group of type (i), we shall call the unique constant c of Theorem (1.10) corresponding to it the angular velocity of the group. This terminology is suggested by the description of c in that theorem.

(1.13) **Corollary.** *If $\{\phi_t\}$, $t \in \mathbf{R}$, is a group of type* (i), *then its invariance polynomial can be written in the form*

$$\phi_1(0,z) = -ic(1 - |\tau|^2)^{-1}[\bar{\tau}z^2 - (1 + |\tau|^2)z + \tau], \tag{1.14}$$

for all $z \in \mathbf{C}$, where τ is the unique point of D left fixed by every ϕ_t for $t \in \mathbf{R}$, and c is the angular velocity of $\{\phi_t\}$, $t \in \mathbf{R}$.

Proof. By (1.6) and (1.11).

2. Hermitian operators and groups of isometries in H^p. For ease of reference we record in (2.1) Forelli's theorem [4, Theorem 2] for $1 \leq p < \infty$, $p \neq 2$, adding [8, Corollary, p. 147] for $p = \infty$ (in the first assertion of (2.1)) and obvious facts for $p = 2$ and $p = \infty$ in the second assertion.

(2.1) **Proposition.** *If T is a linear isometry of H^p onto H^p, $1 \leq p \leq \infty$, $p \neq 2$, then there are $\phi \in \mathscr{M}$ and $\gamma \in C$ such that*

$$(Tf)(z) = \gamma[\phi'(z)]^{1/p}f(\phi(z)) \quad \textit{for all } f \in H^p,\ z \in D. \tag{2.2}$$

On the other hand, if $1 \leq p \leq \infty$, and if γ is a unimodular constant and $\phi \in \mathscr{M}$, then (2.2) *defines a linear isometry T of H^p onto H^p.*

Of course there is a certain amount of indeterminacy in the expression $[\phi'(z)]^{1/p}$ occurring in (2.2) when p is not 1 or ∞ (we take $1/p = 0$ in the case $p = \infty$). However, as is evident from (1.1), ϕ' has a continuous logarithm on $\overline{D}$ which is analytic on D, and any two such logarithms differ on $\overline{D}$ by a constant multiple of $2\pi i$. The understanding is that in the case of any p, $1 \leq p \leq \infty$, such a logarithm W is to be selected, and $[\phi']^{1/p}$ taken to be $\exp[(1/p)W]$ on $\overline{D}$. Changing from one such logarithm to another when $1 < p < \infty$ necessitates only an adjustment of γ while the general form of (2.2) remains the same. For each nonnegative integer n let us define e_n in H^∞ by $e_n(z) = z^n$. It is implicit in the foregoing discussion that in (2.2) the isometry T does not uniquely determine γ or $[\phi']^{1/p}$. However, T does uniquely determine ϕ, since, for all $z \in D$, $\phi(z) = (Te_1)(z)/[(Te_0)(z)]$.

Suppose now that $\{T_t\}$, $t \in \mathbf{R}$, is a strongly continuous one-parameter group of isometries on H^p, $1 \leq p \leq \infty$. Suppose further that for each $t \in \mathbf{R}$ T_t has a representation of the form (2.2) for a (uniquely determined) $\phi_t \in \mathcal{M}$, and appropriately chosen $[\phi_t']^{1/p}$ and unimodular constant. Of course if $p \neq 2$ the latter assumption about the isometries T_t is automatically satisfied, but we are including in our discussion those groups on H^2 for which it holds. From the uniqueness of ϕ_t for each t and the group structure of $\{T_t\}$, $t \in \mathbf{R}$, it is easy to see that $\{\phi_t\}$, $t \in \mathbf{R}$, is a one-parameter group of Möbius transformations of D. Moreover, since for $t \in \mathbf{R}$, $z \in D$, $\phi_t(z) = (T_te_1)(z)/[(T_te_0)(z)]$, the denominator never vanishing, and since evaluation at any $z \in D$ is a continuous linear functional on H^p, it follows that $\phi_t(z)$ is continuous in t for each $z \in D$. In case $\{\phi_t\}$, $t \in \mathbf{R}$, is not a constant group, we shall, for convenience, denote its invariance polynomial by R. We use R to define a (closed) operator $\mathcal{R}$ in H^p as follows:

(2.3) **Definition.** The domain of $\mathcal{R}$, $\mathfrak{D}(\mathcal{R})$, consists of all $f \in H^p$ such that the function on D, $Rf' + (1/p)R'f$, is in H^p. The function defined by this sum is then taken to be $\mathcal{R}(f)$.

(2.4) **Theorem.** *With notation as in the preceding paragraph, let iA be the infinitesimal generator of $\{T_t\}$, $t \in \mathbf{R}$. Then*:

(i) *if $\{\phi_t\}$, $t \in \mathbf{R}$, is a constant group, $A = \alpha I$, for some $\alpha \in \mathbf{R}$, where I is the identity operator on H^p*;

(ii) *if $p < \infty$ and $\{\phi_t\}$, $t \in \mathbf{R}$, is not a constant group, then A is unbounded, and there is a unique real number β such that $\mathcal{R}$ is an extension of $i(A - \beta I)$.*

Proof. The first assertion is easily seen. We begin the proof of (ii) by observing that if $p = 1$ (so that there is no ambiguity in (2.2)), then there is a real constant β such that

$$(2.5) \qquad (T_tf)(z) = e^{i\beta t}\phi_t'(z)f(\phi_t(z)) \quad \text{for } t \in \mathbf{R}, f \in H^1, z \in D.$$

We next develop an analogue of (2.5) when $1 < p < \infty$ which will hold for all t in some appropriate real neighborhood of 0. Let $\mathbf{C}_d$ denote the complex plane with the negative real axis and 0 deleted, and let Log be the principal logarithm on $\mathbf{C}_d$, i.e., $\operatorname{Log} z = \log|z| + i \arg z$, where $|\arg z| < \pi$. Further, for each real number u define F_u on $\mathbf{C}_d$ by the equation $F_u(z) = \exp(u \operatorname{Log} z)$. Then $F'_u = uF_{u-1}$. Also, if $\operatorname{Re}(z_1) > 0$ and $\operatorname{Re}(z_2) > 0$, then $F_u(z_1 z_2) = F_u(z_1)F_u(z_2)$. Since the functions $t \mapsto a_t$ and $t \mapsto b_t$ of (1.2) are continuous (by Theorem (1.3)), it is clear that $\phi'_t \to 1$ uniformly on $\bar{D}$ as $t \to 0$. Thus there is a $\delta > 0$ such that for $t \in (-\delta, \delta)$, $\operatorname{Re}(\phi'_t) > 0$ on $\bar{D}$. For $|t| < \delta$, $[\phi'_t]^{1/p}$ in (2.2) may and will be taken to be $F_{1/p}(\phi'_t)$. In particular $[\phi'_t(0)]^{1/p}$ is differentiable on $(-\delta, \delta)$. Further, with this standardization of $[\phi'_t]^{1/p}$, T_t now has a unique representation in the form (2.2); the unimodular constant will be denoted by γ_t. From the way things have been arranged we have that for t, $u \in (-\delta/2, \delta/2)$, $\gamma_{t+u} = \gamma_t \gamma_u$. Since iA is densely defined, there is $g \in \mathfrak{D}(iA)$ such that $g(0) \neq 0$. For $|t| < \delta$, $(T_t g)(0) = \gamma_t [\phi'_t(0)]^{1/p} g(\phi_t(0))$. Let us rewrite the function $t \mapsto \gamma_t$ as $\gamma(\cdot)$. The functions (of t) $(T_t g)(0)$ and $g(\phi_t(0))$ are differentiable at $t = 0$, the former because $g \in \mathfrak{D}(iA)$, the latter by Theorem (1.3). Since $g(0) \neq 0$, the above equation for $(T_t g)(0)$ now yields differentiability of $\gamma(\cdot)$ at 0. Thus for $t \in (-\delta/2, \delta/2)$,

$$\lim_{h \to 0} h^{-1}[\gamma(t+h) - \gamma(t)] = \gamma(t) \lim_{h \to 0} h^{-1}[\gamma(h) - 1] = \gamma(t)\gamma'(0).$$

The existence of a real constant β such that $\gamma(t) = e^{i\beta t}$ on $(-\delta/2, \delta/2)$ is an elementary consequence of the relations $\gamma'(t) = \gamma'(0)\gamma(t)$ on $(-\delta/2, \delta/2)$, $\gamma(0) = 1$, $|\gamma(t)| \equiv 1$. We have

$$(2.6) \qquad \begin{aligned} (T_t f)(z) &= e^{i\beta t}[\phi'_t(z)]^{1/p} f(\phi_t(z)) \\ &\qquad \text{for } f \in H^p,\ z \in D,\ t \in (-\delta/2, \delta/2). \end{aligned}$$

If we combine the cases $p = 1$ and $1 < p < \infty$ by applying (2.5) and (2.6) respectively to the strongly continuous one-parameter group of isometries $\{e^{-i\beta t} T_t\}$ (with generator $i(A - \beta I)$), we get for all $f \in \mathfrak{D}(iA)$, all $z \in D$:

$$\begin{aligned} [i(A - \beta I)f](z) &= d/dt\,|_{t=0}\,[(e^{-i\beta t} T_t f)(z)] \\ &= d/dt\,|_{t=0}\,\{[\phi'_t(z)]^{1/p} f(\phi_t(z))\} = (\mathscr{R} f)(z). \end{aligned}$$

The uniqueness of β as described in (ii) is evident. Finally, if A were bounded on its domain, then $i(A - \beta I)$ would also be. But the latter, being an infinitesimal generator, is closed and densely defined. Hence we would get that $\mathscr{R}$ is a bounded operator defined on all of H^p, an absurd conclusion, since, for instance, $\|\mathscr{R} e_n\| \geq n\|R\| - (1/p)\|R'\|$, whereas R is not the zero polynomial.

Definition. For $1 \leq p < \infty$, $p \neq 2$, an unbounded hermitian operator A has associated with it as above a uniquely determined nonconstant group $\{\phi_t\}$. We shall say that A (or the group $\{T_t\}$ generated by iA) is of type (i), (ii), or (iii) according as $\{\phi_t\}$ is. If $p = 2$ and A is an unbounded hermitian operator such that each T_t in the group generated by iA is representable in the form (2.2), then the observations in the preceding sentence still apply to A, and the same terminology regarding type will be used.

The next theorem was pointed out by L. A. Rubel, to whom the authors wish to express their appreciation.

(2.7) **Theorem.** *Every hermitian operator in H^∞ is bounded, and is in fact a real scalar multiple of the identity operator I on H^∞.*

Proof. Let A be a hermitian operator in H^∞, and let iA generate the strongly continuous group $\{T_t\}$, $t \in \mathbf{R}$. It is easy to see from (2.1) that because $p = \infty$ there is a real constant γ such that $(T_t f)(z) = e^{i\gamma t} f(\phi_t(z))$, for $f \in H^\infty$, $t \in \mathbf{R}$, $z \in D$. It suffices to show that the strongly continuous group of isometries $\{e^{-i\gamma t}T_t\}$,, $t \in \mathbf{R}$, (with generator $i(A - \gamma I)$) is constant, or in other words the group $\{\phi_t\}$, $t \in \mathbf{R}$, is constant. For $w \in C$, define g_w on D by $g_w(z) = \exp[(1 + z\overline{w})/(z\overline{w} - 1)]$. It is well known and elementary that: (i) $g_w \in H^\infty$; (ii) $|g_w(z)| \to 1$ as z approaches any point on C distinct from w; and (iii) $|g_w(z)| \to 0$ as z approaches w nontangentially. Fix w, and let t_0 be any real number. We claim there is a $\delta > 0$ such that $\phi_t(w)$, as a function of t, is constant in the open interval $(t_0 - \delta, t_0 + \delta)$. Otherwise for each positive integer n, there is $t_n \in \mathbf{R}$ with $|t_n - t_0| < n^{-1}$ and $\phi_{t_n}(w) \neq \phi_{t_0}(w)$. For each n and each $z \in D$, $\|T_{t_n} g_w - T_{t_0} g_w\| \geq |g_w(\phi_{t_n}(z))| - |g_w(\phi_{t_0}(z))|$. If we let z approach $\phi_{-t_0}(w)$ nontangentially, we get that $\|T_{t_n} g_w - T_{t_0} g_w\| \geq 1$, contradicting the strong continuity of $\{T_t\}$. Clearly $t \mapsto \phi_t(w)$ is a continuous function on $\mathbf{R}$. By the connectedness of $\mathbf{R}$, $\phi_t(w) = w$ for all $t \in \mathbf{R}$. Since w was an arbitrary point of C, we are done.

The next theorem is an obvious consequence of (2.4) and (2.7) which we state in a form that contrasts H^2 with all the other H^p Banach spaces.

(2.8) **Theorem.** *If $1 \leq p \leq \infty$, $p \neq 2$, then the bounded hermitian operators on H^p are precisely the real scalar multiples of the identity operator on H^p.*

(2.9) **Examples.** It is easy to see that if $\{\phi_t\}$, $t \in \mathbf{R}$, is a one-parameter group of Möbius transformations of type (i), (ii), or (iii), then the equation $(T_t f)(z) = \phi_t'(z) f(\phi_t(z))$, for all $f \in H^1$, $t \in \mathbf{R}$, $z \in D$, defines a strongly continuous one-parameter group of isometries (and implicitly a hermitian operator) in H^1 of type (i), (ii), or (iii), respectively. Appropriate groups $\{\phi_t\}$, $t \in \mathbf{R}$, were described in (1.9) and (1.10). In order to provide examples of the discussion in §3 and to illustrate the contrast between H^∞ and the other Hardy spaces, we shall produce some (automatically unbounded) hermitian operators of type (i) in the spaces H^p for $1 \leq p < \infty$. It is easy to see that there is a positive number δ such that if τ is any complex number of modulus less than δ and c is any nonzero real number, then the group $\{\phi_t\}$ of type (i) constructed from c and τ by the equations in (1.11) has the property that $\operatorname{Re}(e^{-ict}\phi_t'(z)) > 0$ for all $t \in \mathbf{R}$ and all $z \in \overline{D}$. Let such a τ and c be selected. For the corresponding group $\{\phi_t\}$ of type (i), we take $[\phi_t']^{1/p}$ to be (in the notation of the proof of Theorem (2.4)) $[\exp(ict/p)]F_{1/p}(e^{-ict}\phi_t')$. One verifies that the equation $(T_t f)(z) = [\phi_t'(z)]^{1/p} f(\phi_t(z))$ for all $f \in H^p$, $t \in \mathbf{R}$, $z \in D$, defines a strongly continuous one-parameter group of isometries (and corresponding hermitian operator) of type (i) in H^p. We further observe that if α is any real number, then the strongly continuous one-parameter group $\{S_t\}$, $t \in \mathbf{R}$, where $S_t = e^{i\alpha t}T_t$ for all t, is also of type (i). In particular if we take $\tau = 0$, and $\alpha = -c/p$, then the group $\{S_t\}$ acquires the form $(S_t f)(z) = f(e^{ict}z)$, for all $f \in H^p$, $z \in D$, $t \in \mathbf{R}$.

3. **Spectral theory of hermitian operators of type** (i). It is a standard fact [2, p. 656] that if K is a closed operator in a Banach space whose resolvent set $\rho(K)$ is nonvoid, then the corresponding resolvent operator $R(\cdot; K)$ has a value which is a compact operator if and only if each of its values is a compact operator. In this case we shall say that K has compact resolvent. It is immediate from the definition that a hermitian operator in a Banach space is always closed and densely defined with spectrum contained in $\mathbf{R}$. We shall see presently that a hermitian operator of type (i) in H^p always has compact resolvent.

Throughout this section A will be a hermitian operator of type (i) in H^p, $1 \le p < \infty$. The notation will be as in the statement of Theorem (2.4). In particular $\{T_t\}$ and $\{\phi_t\}$ will be the groups corresponding to A. We denote by τ the unique point of D left fixed by every ϕ_t, and by c the angular velocity of $\{\phi_t\}$. If $\tau \neq 0$, then the function $(z - \bar{\tau}^{-1})$ has of course a continuous logarithm Q on $\bar{D}$ which is analytic on D. We choose such a function Q (it will be immaterial which one) and fix it for the rest of this section. For any real constant u define the function $(z - \bar{\tau}^{-1})^u$ on $\bar{D}$ to be $\exp(uQ)$.

(3.1) **Theorem.** *If A is a hermitian operator of type* (i) *in H^p, $1 \le p < \infty$, then*:

(i) *there is a unique real number β such that $A = \beta I - i\mathcal{R}$, where $\mathcal{R}$ is given by* (2.3);

(ii) *the eigenvalues of A are precisely the real numbers $\lambda_n = c(n + p^{-1}) + \beta$, $n = 0, 1, 2, \ldots$*;

(iii) *for each n the eigenmanifold of A corresponding to λ_n is the one-dimensional span of the function $f_{n,\tau}$, where, for each $z \in D$,*

$$f_{n,\tau}(z) = (z - \tau)^n/(z - \bar{\tau}^{-1})^{n+(2/p)}, \qquad \text{if } \tau \neq 0,$$
$$= z^n, \qquad \text{if } \tau = 0;$$

(iv) *A has compact resolvent, and hence pure point spectrum.*

Proof. By Theorem (2.4)(ii) there is a unique real number β_0 such that $\mathcal{R}$ is an extension of $i(A - \beta_0 I)$. Assertion (i) is obviously equivalent to the equation $i(A - \beta_0 I) = \mathcal{R}$. Since $i(A - \beta_0 I)$ generates a strongly continuous one-parameter group of isometries, its spectrum is contained in the imaginary axis. If we show the existence of a complex number γ with nonzero real part such that γ is not an eigenvalue of $\mathcal{R}$, then $\gamma I - \mathcal{R}$ will be one-to-one and extend $\gamma I - i(A - \beta_0 I)$, which maps $\mathfrak{D}(A)$ onto H^p. Hence assertion (i) will be demonstrated. We show in fact that the eigenvalues of $\mathcal{R}$ are precisely the complex numbers $\mu_n = ic(n + p^{-1})$, $n = 0, 1, 2, \ldots$, and that the eigenmanifold of $\mathcal{R}$ corresponding to μ_n is the one-dimensional span of $f_{n,\tau}$. This will establish assertions (i), (ii), and (iii). Application of Theorem (1.7) and Corollary (1.13) gives the conclusion that if $\tau \neq 0$ (resp., $\tau = 0$), then the invariance polynomial R has the form $R(z) = -\bar{\tau}ic(1 - |\tau|^2)^{-1}(z - \tau)(z - \bar{\tau}^{-1})$ for all $z \in \mathbf{C}$ (resp., $R(z) = icz$ for $z \in \mathbf{C}$). If λ is an eigenvalue of $\mathcal{R}$ with corresponding eigenvector h, then we can choose a counterclockwise circular contour Γ concentric with C and contained in D such that Γ encloses τ and h never vanishes on Γ. The equation $(\mathcal{R}h)(z) = \lambda h(z)$, $z \in D$, rearranges on Γ to give $h'h^{-1} = (\lambda - p^{-1}R')R^{-1}$. If we take the contour integral around Γ of each side of this last equation, the argument principle shows that λ must be μ_n for some n. Next one verifies that on the region obtained by deleting τ from D the differential equation $f' + (1/R)(p^{-1}R' - \mu_n)f = 0$ has an integrating factor given by $[(z - \bar{\tau}^{-1})^{n+(2/p)}/(z - \tau)^n]$ or by z^{-n} according as $\tau \neq 0$ or $\tau = 0$. We conclude easily that μ_n is an eigenvalue of $\mathcal{R}$ with corresponding eigenmanifold spanned by $f_{n,\tau}$. It remains only to prove (iv). Since a closed operator with nonempty resolvent set and compact resolvent always has only eigenvalues in its spectrum [7, Theorem 5.14.2], it suffices to show that A (or, equivalently, $\mathcal{R}$) has compact resolvent. To motivate the next step we remark that if g is an arbitrary function in H^p and λ is an unspecified complex parameter, then examination of the differential equation $(1/R)(\mathcal{R} - \lambda)f = (g/R)$ for *formal* integrating factors (involving complex powers of $(z - \tau)$, etc.) leads to a convenient situation if λ is taken to be $ic(-1 + p^{-1})$. However, rather than argue the next step in this fashion, we shall simply verify the outcome of this procedure, and go on to conclude the proof by showing that $[\mathcal{R} - ic(-1 + p^{-1})I]$ maps the domain of $\mathcal{R}$ one-to-one onto H^p and has a compact inverse. For convenience put $\alpha = ic(-1 + p^{-1})$. For $g \in H^p$ it is straightforward to check that if $\tau \neq 0$ (resp., if $\tau = 0$), then the function $f(z)$ given by

$$(\bar{\tau}ic)^{-1}(|\tau|^2 - 1)(z - \bar{\tau}^{-1})^{1-(2/p)}(z - \tau)^{-1}\int_\tau^z g(\xi)(\xi - \bar{\tau}^{-1})^{(2/p)-2}d\xi$$

(resp., by $(icz)^{-1}\int_0^z g(\xi)d\xi$), which has a removable singularity at τ, satisfies throughout D the differential equation $Rf' + (p^{-1}R' - \alpha)f = g$. Moreover, as one easily sees, in order for f to be in H^p it is sufficient that

$$\sup_{0<L<1}\int_0^{2\pi}\left|\int_0^L g(re^{i\theta})(re^{i\theta} - \bar{\tau}^{-1})^{(2/p)-2}dr\right|^p d\theta$$

(resp., $\sup_{0<L<1}\int_0^{2\pi}|\int_0^L g(re^{i\theta})dr|^p d\theta$) be finite in the case $\tau \neq 0$ (resp., $\tau = 0$), which is an immediate consequence of the generalized Minkowski inequality [14, I.9.12, p. 19] (since g is in H^p). Thus $\mathcal{R} - \alpha I$ maps $\mathfrak{D}(\mathcal{R})$ onto H^p and is one-to-one (since $\alpha \neq \mu_n$, $n = 0, 1, 2, \ldots$). A standard application of the closed graph theorem now shows that $(\mathcal{R} - \alpha I)^{-1}$ is continuous on H^p and α is in the resolvent set of $\mathcal{R}$. By use of the formula $(\mathcal{R} - \alpha I)^{-1}g = f$, as above, we shall show that $(\mathcal{R} - \alpha I)^{-1}$ is a compact operator on H^p. Accordingly, let $\{g_n\}_{n=1}^\infty$ be any bounded (in H^p) sequence of polynomials. The proof of the theorem will be complete once it is established that $\{(\mathscr{R}-\alpha I)^{-1}g_n\}_{n=1}^{\infty}$ has a Cauchy subsequence in H^p. However, by passage to boundary-value functions on C, the existence of such a subsequence is seen to be a consequence of the following lemma.

(3.2) **Lemma.** *Suppose that $\{G_n\}_{n=1}^{\infty}$ is a sequence of functions analytic on some disc $\{z \in \mathbf{C}: |z| < \eta\}$, where $\eta > 1$. Suppose further that for some p, $1 \le p < \infty$, the restrictions of the functions G_n to C form a bounded sequence in H^p. For each n define the function h_n on C by setting $h_n(z) = \int_0^z G_n(\xi)\,d\xi$ for each $z \in C$. Then $\{h_n\}_{n=1}^{\infty}$ has a Cauchy subsequence in H^p.*

Proof. For each n let G_n have the power series expansion $G_n(z) = \sum_{k=0}^{\infty} a_k^{(n)} z^k$ for $|z| < \eta$. Clearly for each $z \in C$, $h_n(z) = \sum_{k=0}^{\infty} a_k^{(n)}(k+1)^{-1}z^{k+1}$, the series converging uniformly on C. Since the sequence $\{G_n\}$ restricted to C is bounded in H^p, the double sequence $\{a_k^{(n)}\}_{k=0,n=1}^{\infty}$ is bounded. If $1 \le p \le 2$, then, for all m, n

$$(3.3) \qquad \|h_m - h_n\|^2 \le \int |h_m - h_n|^2 d\sigma = \sum_{k=0}^{\infty} |a_k^{(m)} - a_k^{(n)}|^2(k+1)^{-2}$$

where σ is normalized Lebesgue measure on C. If $2 < p < \infty$, then let q be the conjugate index to p (i.e., $p^{-1} + q^{-1} = 1$). By [3, Theorem 6.1] we have, for all m, n,

$$(3.4) \qquad \|h_m - h_n\|^q \le \sum_{k=0}^{\infty} |a_k^{(m)} - a_k^{(n)}|^q(k+1)^{-q}.$$

However, if r is any real number greater than 1, then the sequence of vectors $\{x_n\}$ in l^r given by $x_n = \{a_k^{(n)}(k+1)^{-1}\}_{k=0}^{\infty}$ is obviously bounded in l^r, and furthermore $\lim_{N\to\infty} \sum_{k=N}^{\infty} |a_k^{(n)}|^r(k+1)^{-r} = 0$ uniformly in n. It follows that $\{x_n\}$ has a Cauchy subsequence in l^r, and using this fact in (3.3) and (3.4) completes the proof of the lemma.

Remarks. (i) It is easy to see that for a type (i) hermitian operator A the domain $\mathfrak{D}(\mathscr{R})$ of the operator $\mathscr{R}$ above consists of all $f \in H^p$ such that $f' \in H^p$. In particular, by [3, Theorem 3.11], $\mathfrak{D}(\mathscr{R}) \subseteq H^{\infty}$. Thus once it was shown that the function $f = (\mathscr{R} - \alpha I)^{-1}g$ in the proof of Theorem (3.1) was in H^p, it followed that f was in H^{∞}. However, we did not need to make use of this circle of ideas. (ii) It is clear that for a given A of type (i) the eigenfunctions $f_{n,\tau}$, $n = 0, 1, 2, \ldots$, span a dense linear manifold in H^p. (iii) An operator A of type (i) can be completely recovered from the collection of all its eigenvectors together with its spectrum. This can be seen, for example, by observing the following facts. τ is the only point of D which occurs as a zero of an eigenvector of A. Neighboring points (in $\mathbf{R}$) of the spectrum of A are spaced apart by a constant distance equal to $|c|$. In any enumeration the points of the spectrum of A form a sequence which approaches $(\operatorname{sgn} c)\,\infty$, where $\operatorname{sgn} c$ is the sign of c. If $c > 0$ (resp., if $c < 0$) then the minimum (resp., the maximum) point in the spectrum of A is $cp^{-1} + \beta$.

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